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HOW TO DESCRIBE PURE FORM AND HOW TO MEASURE DIFFERENCES IN SHAPES USING SHAPE NUMBERS

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Abstract – The *shape number* of a curve is derived for two-dimensional non-intersecting closed curves that are the boundary of simply connected regions. This description is independent of their size, orientation and position, but it depends on their shape. Each curve carries "within it" its own shape number. The *order* of the shape number indicates the precision with which that number describes the shape of the curve. For a curve, the order of its shape number is the length of the perimeter of a 'discrete shape' (a closed curve formed by vertical and horizontal segments, all of equal length) closely corresponding to the curve. A procedure is given that deduces, without table look-up, string matching or correlations, the shape number of any order for an arbitrary curve. To find out how close in shape two curves are, the *degree* of similarity between them is introduced; dissimilar regions will have a low degree of similarity, while analogous shapes will have a high degree of similarity. Informally speaking, the degree of similarity between the shapes of two curves tells how deep it is necessary to descend into a list of shapes, before being able to differentiate between the shape of those two curves. Again, a procedure is given to compute it, without need for such list or grammatical parsing or least square curve or area fitting. The degree of similarity maps the universe of curves into a tree or hierarchy of shapes. The *distance* between the shapes of any two curves, defined as the inverse of their degree of similarity, is found to be an ultradistance over this tree. The shape number is a description that changes with skewing, anisotropic dilation and mirror images, as the intuitive psychological concept of "shape" demands. Nevertheless, at the end of the paper a related Theory "*B*" of shapes is introduced that allows anisotropic changes of scale, thus permitting for instance a rectangle and a square to have the same *B* shape. These definitions and procedures may facilitate a quantitative study of shape.

Curve description · Chain encoding · Shape code · Silhouettes · Shape numbers · Form similarity · Shape comparison · Measure of shape difference · Binary picture · Image processing

INTRODUCTION

The study of shape is an important part of the field of Pattern Recognition.

As pointed out by Lord Kelvin, a science begins to emerge when it is possible to make measurements of the phenomena that such science seeks to understand, allowing quantitative comparison and mathematical relations among them.

This paper gives a procedure to measure (i.e. to assign a number to) the resemblance between any two shapes.

With the help of procedures like this, a quantitative study of shape may be possible.

Previous work on shape

Shape extraction is an active field. Sequential extraction of shape features[4] can be performed making only one pass over the image. For global shape analysis, several authors have used Freeman chains, medial axis transforms, decomposition into primary convex subsets, polar co-ordinates,[6] decomposition at concave vertices; decomposition by clustering, mirroring axes [7] and stroke detectors. These and other methods are reviewed by Pavlidis.[5]

1. THE SHAPE NUMBERS

1.1 *What is a shape*

A *region* is a simply connected portion of a plane limited by a curve boundary. That is, no holes; no self-intersecting boundary. It is a closed boundary. A given region has a size, a position, and an orientation in the plane. This defines a flat region, which is uniquely defined by the curve it has as boundary. This paper deals with shapes of regions, but the shape numbers used here can also be applied to open curves. In addition, Section IV describes regions with holes.

A *shape* is what remains of a region after disregarding its size, position and orientation in the plane. That is, two regions have the same shape if we can make them coincide exactly by translation and rotation in the plane, in addition to a uniform change of scale (the

x and y co-ordinates increase by the same factor).

A region and its mirror images will not have the same shape, in general.

This definition coincides with the intuitive psychological definition of "shape".

If a notation is going to be used to represent the shape of a region, it has to be independent of the position, orientation and size of such region. It should be reproducible: a region, when translated, magnified and rotated should still give the same description as when untransformed. Two regions with different shapes should produce different descriptions. Finally, the shape number should be unique for a given region; for instance, it should not depend on an arbitrary starting point or a particular co-ordinate system.

If the notation can be deduced exclusively from the region, without comparison with a table of canonical shapes or shape descriptors, for instance, then we can expect savings in memory and computer time for the procedure that computes the shape description.

1.2 *Continuous and discrete shapes*

A shape is *discrete* if the boundary of the region is formed by segments of a square grid. Otherwise, the shape is *continuous*.

1.3 *Matting a continuous shape into a discrete shape*

A square grid may be overlaid on top of a continuous shape to obtain a discrete shape. The quantization of the shape is as follows: a square of the grid is "black" (inside the discrete shape) if more than 50% of it is covered by the continuous shape; otherwise it is "white" or outside (Fig. 1). The size, orientation and position of this grid will influence the resulting discrete shape.

A discrete shape, obtained from a continuous shape in the above manner, can not be a shape descriptor of the continuous shape, because it depends on the size and orientation of the grid. This will be solved in Section 1.6.

Now, some shape descriptors will be given.

1.4 *Eccentricity*

The *eccentricity* (ratio of the major to minor axis, Fig. 2) of a region is a descriptor that depends only on its shape.

The *major axis* of a region is the line joining the two perimeter points furthest away from each other. The *minor axis* is perpendicular to the major axis, and of length such that a box could be formed that just

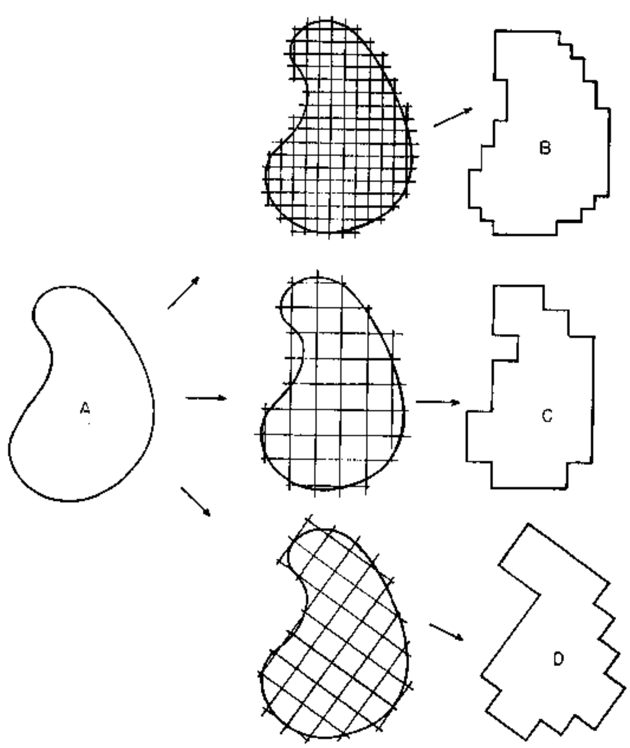


Fig. 1. Continuous and discrete shapes. Continuous shape A gives rise to several discrete shapes B, C, D. If it is desired to have a *unique* discrete shape derived from A, then it is necessary to specify the grid size (related to the *order* of the discrete shape), as well as its orientation and position with respect to the continuous curve A. In this manner, for a given order n, the discrete shape corresponding to A will be unique. This is accomplished in Section 1.6.

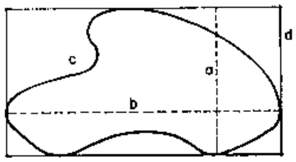


Fig. 2. Basic rectangle. (a) Minor axis of c. (b) Major axis of c. (c) Region. (d) Basic rectangle of c. The eccentricity $e = b/a$ is always greater than or equal to 1. It is a shape descriptor, although not a good one.

encloses the region. This box is called the *basic rectangle* (Fig. 2).

Occasionally, there will be more than one major axis in a region. In that case, select that which gives the shorter minor axis; if necessary, add additional criteria to make the choice of major axis a unique choice.

1.5 *Freeman chain and its derivative*

Freeman chain in four directions. For a given region and a given square grid of fixed orientation and size, the Freeman chain in four directions is the curve obtained by walking clockwise on the grid (on its "wires") around and outside the squares that are more than half contained by the region (Fig. 3).

Derivative of Freeman Chains. It is the chain number obtained by clockwise replacing each convex corner of the Freeman chain by a 1, each straight corner by a 2, and each concave corner by a 3, as Fig. 3 suggests. The number obtained ("F" in Fig. 3) will be different if we change the size or orientation of the grid. In the next section a method appears that makes the "derivative of Freeman chain" independent of these changes. This new derivative will be called the shape number.

1.6 *The shape numbers*

This section tells how to obtain our proposed description for the shape of *shapes* and *regions*. The procedure to find the shape number of a region is as follows:

1. A grid of arbitrary cell size is overlaid on top of the region. A "black" region is formed with all the cells that fall 50 per cent or more inside the region.

2. The boundary of such a black region is the chain sought after. This chain is denoted by its derivative notation (q.v.). We collect these numbers travelling clockwise. Refer to Fig. 4.

Observe that there are several strings of digits 1, 2 and 3 corresponding to the above chain, depending on the starting point (see Fig. 4):

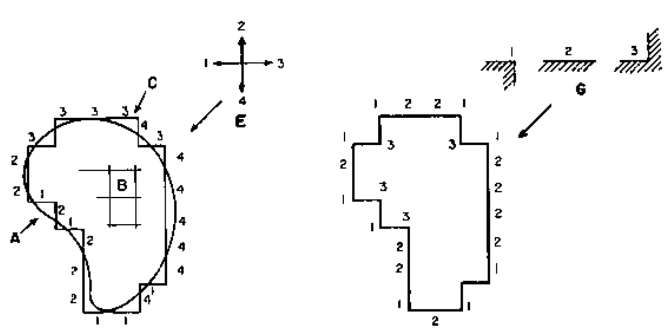
12131131213113	(A)
21311312131131	(B)
13113121311312	(C)
31131213113121	(D)
11312131131213	(E)
13121311312131	(F)
31213113121311	(G)
12131131213113	(H)
21311312131131	(I)
13113121311312	(J)
31131213113121	(K)
11312131131213	(L)
13121311312131	(M)
31213113121311	(N)

Observe also that one of them is a minimum, when regarded as a number in base 3: (E) in the above example.

3. Select the chain that is minimum as the chain that represents the region. In the example, it is 11312131131213. Observe that the minimum chain always starts with a 1, since every discrete shape contains at least four 1's.

What size of grid? What orientation? Unless a procedure is given that normalizes these questions and provides unique answers, a region will have several shape numbers.

The adopted posture is that the orientation of the grid will be normalized, but its size will be a parameter that will allow us to vary the precision of the shape number. Nevertheless, although the size of the cell of the grid varies according to the precision, the number of segments of the grid (sides of each cell) into which the region will be mapped is no longer at user's will, but



D : 223233343444441411222121

F : 121312213122221312122313

Fig. 3. Chains. A: the region. B: The grid. C: The Freeman chain in four directions. D: Its chain number. E: The four directions of (B) used to code (C) into (D). F: the derivative of (C). G: The three types of corners used to code (C) into (F).

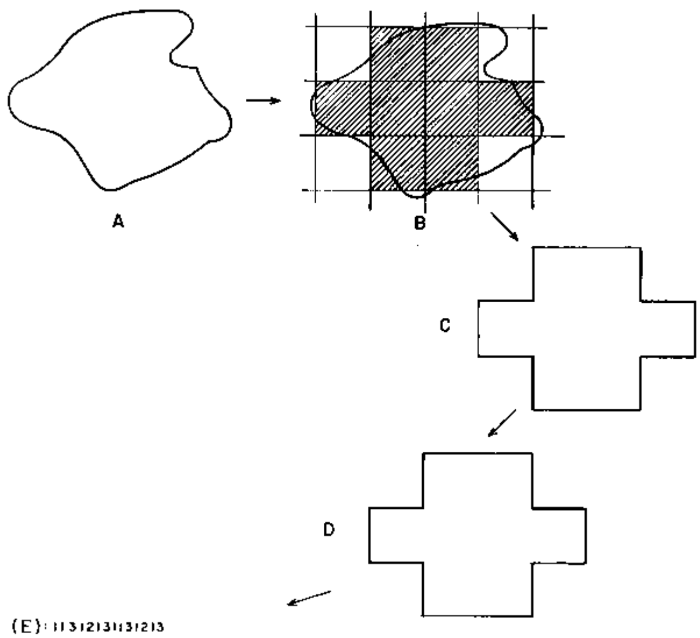


Fig. 4. Shape number. (*A*) the continuous shape. (*C*) The discrete shape. (*E*) The shape number.

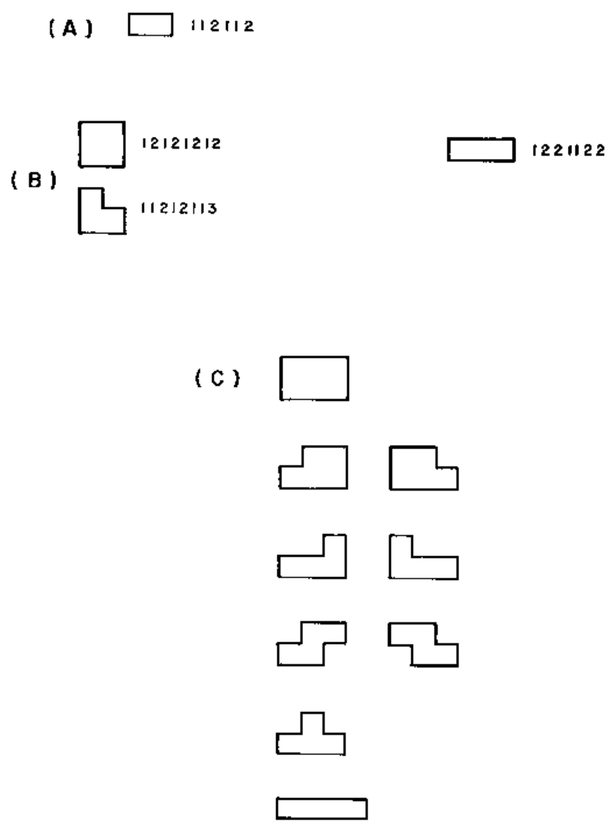


Fig. 5. All the shapes of orders 6, 8 and 10. (*A*) Of order 6. (*B*) Of order 8. (*C*) Of order 10.

it is dictated by the precision he specifies.

The orientation of the grid is not arbitrary, but it is made to coincide with the major axis of the region. The reason is clear: each region should carry along with it its own direction of the grid. In this manner, if the region rotates, the grid rotates the same amount and a code is obtained invariant under rotations.

Procedure to achieve a unique shape number. Given a region surrounded by its basic rectangle, a grid of a given (fixed) size could be placed on top of the rectangle, in order to extract the unique shape number of the region. Instead, the user is allowed to tell how many digits he wants his shape number to contain. That is known as the *order* of the shape number.

It is clear that the same shape gives rise to several shape numbers. But, given n, the shape number of order n of that shape is unique.

Shortly a procedure will be shown to find the shape number of order n of a region, for a given n. Before that, however, the families of discrete closed shapes of several orders are presented.

All the shapes of order 4. These are all the regions that can be formed with four sticks of the same size, when they can be placed only collinearly or at 90 degrees with respect to each other.

There is only one closed shape of order 4, the square: 1111.

This is the most primitive or fundamental shape. Imagine you are looking at things very far away; you can not really differentiate much. All objects would look round (square, in this paper) and equal.

All the shapes of order 5. No shape number of odd order represents a closed figure. For a closed figure,

number of corners = number of sticks = order of figure.

This paper does not deal with open figures. Not all ternary numbers with an even number of digits are shape numbers. Most of them do not close.

All the shapes of order 8, 10 *and* 12. See Figs. 5 and 6.

2. USING THE SHAPE NUMBERS FOR SHAPE DESCRIPTIONS

2.1 *The order of a shape number*

The order of a shape number is the number of ternary digits that the shape number contains. It is always even, because the boundary is closed.

2.2 *How to find the shape number of order* n *of a continuous shape*

The procedure is as follows:

1. Find the basic rectangle of the region.

2. From the family of discrete shapes of order n, find the rectangle of order n with eccentricity closest to that of the region. (This is easy. For instance, for $n = 22$, the rectangles of order 22 – those with perimeter equal to 22 – are of sides 6 by 5, 7 by 4, 8 by 3, 9 by 2 and 10 by 1.)

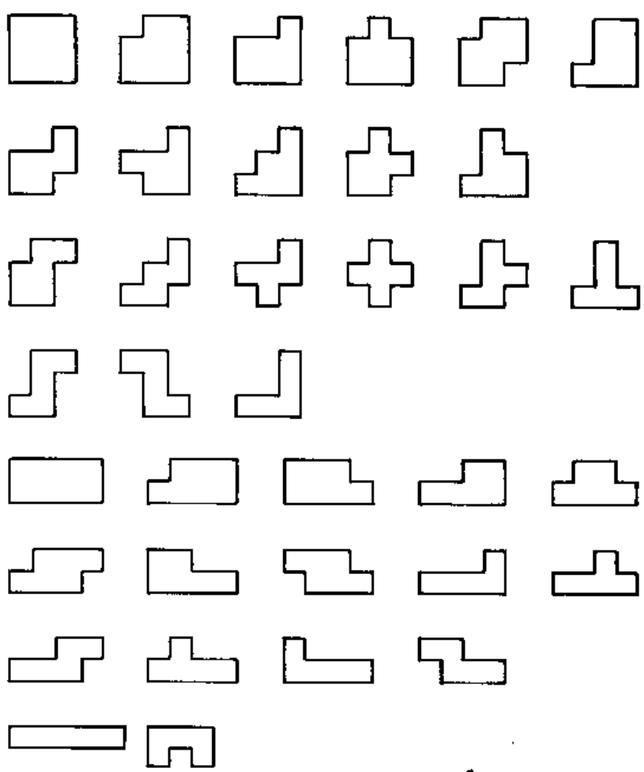


Fig. 6. All the shapes of order 12.

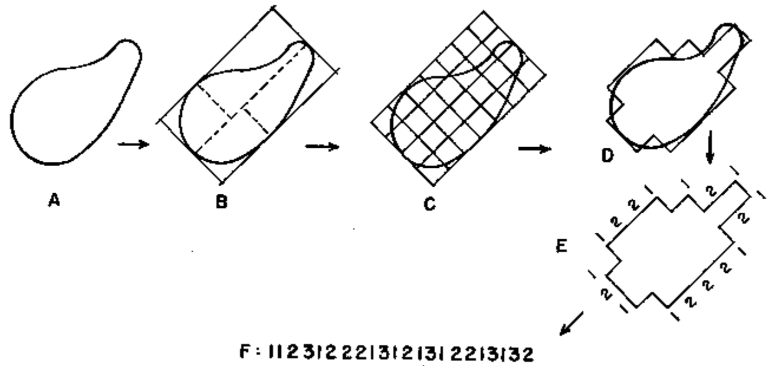


Fig. 7. Finding the shape number of order 22. The continuous (*A*) is encased in its basic rectangle (*B*). Given $n=22$, a square grid of appropriate size (see text) is centered (*C*) on the basic rectangle. A discrete shape (*D*) is obtained. The "derivative notation" (*E*) is found. Traveling clockwise, the chain with the minimum absolute value (*F*) is the shape number of order 22 digits. (*D*) has 22 sides, as well as 22 corners.

In practice, it is better to approximate the longer side of the rectangle instead of the eccentricity. If e = eccentricity, one can deduce that the longer side is $b = (n/2)(e/1+e)$. Select a rectangle with longest side closest to that quantity. Lay this rectangle, centered, so as to cover the region, and make a grid of a by b square cells ("C", in Fig. 7).

3. Make black (= 1) all those cells falling more than 50% inside the region; leave white (0, outside) all others.

The boundary of this black region, expressed in the derivative notation, is the desired shape number.

Remember to write down the digits of the chain traveling clockwise, and selecting as the starting point the corner of type 1 that makes the chain number the smallest of the n possible chain numbers. An example is given in Fig. 7.

Notice that the resulting shape number is indeed of order n. This will not be true if the figure has depressions (concavities) in its boundary. The depression in the boundary makes the order bigger. Each depression of depth k increases the order of the shape by $2k$.

When, looking for a shape number of order n, if a number of order $n+2d$ results, try next to look for a shape number of order $n-2d$. Due to the presence of the holes, the shape number $n-2d$ will be increased by an amount equal to the "hole excess" $2d$, thus yielding the desired order n. This relation holds only approximately, since the size of the holes of order n is smaller than those of order $n-2d$. Thus, in practice, try the basic rectangles or order $n-2d$, $n-2d+2$, $n-2d+4$, $\ldots, n-2$, and when we obtain a shape number of order n, that is it. See also Section 2.3.

Properties of the shape number. It is insensitive to orientation of the region, to its position, to its size, and to the origin of the chain. It is therefore appropriate to think that the shape number of a region indeed describes its *shape* (cf. Section 1.1).

Also, since it is possible to compute the shape number of a region without reference to a table of sorted shapes (canonical shapes), we avoid making correlations or comparisons of shapes or of strings. That is, the shape number of a region can be deduced solely from the region.

In addition, the precision of the resulting shape number can be varied. This is done with the order of the shape number; that is, the size of the sticks (or of the grid) that we use to find it.

2.3 *Shapes without shape numbers*

A shape with a thin isthmus (narrower than the size of the grid) will not yield one shape number, since the procedure of Section 2.2 will split the continuous shape into several discrete shapes (Part II of Fig. 8). Some shapes with depressions in their boundaries may not have a shape number. This is discussed in Section 3.3d.

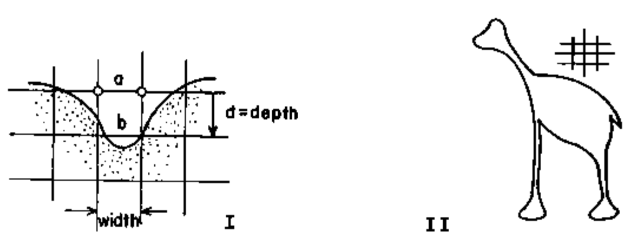


Fig. 8. Depressions and degenerate shapes. I. A depression of depth d increases the shape number by $2d$. II. Degenerate regions split the discrete shape but do not have a shape number.

3. USING THE SHAPE NUMBERS TO MEASURE SHAPE SIMILARITY

3.1 *The degree of similarity between two shapes*

The shape number of a region enables us to find instances of a given shape, even when distorted by enlargement or rotation. It answers the questions "Have these two regions the same shape?", up to an order n.

In practice, however, a shape rarely repeats itself, due to noise and the allowable variations (for instance, ten silhouettes of apples have similar but not identical shapes). The relevant questions to answer are "How much different are these two forms?", "How much do these two shapes resemble each other?", "Is region A closer in shape to B, or to C?". This section gives a procedure to quantitatively answer these questions.

When the shapes of two regions A and B are compared, we can notice that the shape of order 4 of A, $s_4(A)$, is equal to 1111 (the only shape of order 4), and is therefore equal to $s_4(B)$.

Also $s_6(A) = s_6(B)$; probably $s_8(A) = s_8(B)$. It is likely that their first few shape numbers be identical. The reason is that the discrete shapes are coarse and not varied at low orders, where the "resolution" is low.

Nevertheless, most likely $s_{100}(A) \neq s_{100}(B)$, also $s_{98}(A) = s_{98}(B)$, etc. This is expected, because, due to the finer precision at higher orders, there exists a large variety of shapes, thus the discrimination between A and B is more demanding.

Of course, if A and B were very similar (but not identical), one might need to go up to say 170 to find that $s_{170}(A) \neq s_{170}(B)$. On the other hand, if they are visibly different (not alike at all), already at order 10 we will find $s_{10}(A) \neq s_{10}(B)$.

Thus, as we increase the order n of the two shape numbers $s_n(A)$ and $s_n(B)$, they begin equal but at some order they become different. How long they remain equal gives us an idea of the similarity between the shapes of A and B.

Degree of similarity k between the shapes of two regions A and B: it is the largest order for which their shape numbers still coincide.

That is, it is the largest m for which $s_m(A) = s_m(B)$, but $s_{m+i}(A) \neq s_{m+i}(B)$ for all i greater than 0.

That is, we have $s_4(A) = s_4(B)$, $s_6(A) = s_6(B)$, $s_8(A) = s_8(B), \ldots, s_k(A) = s_k(B)$, $s_{k+2}(A) \neq s_{k+2}(B)$, $s_{k+4}(A) \neq s_{k+4}(B), \ldots$.

If A and B are regions with degree k of similarity we write $a \approx_k b$.

Example. For the figures of Fig. 9 we have for A to F:

$s_4(A) = s_4(B) = \cdots = s_4(F) = 1111$;

$s_6(A) = s_6(B) = \cdots = s_6(F) = 112112$;

$s_8(A) = s_8(D) = s_8(E) = 12121212$;

$s_8(B) = 11212113$; $\quad s_8(C) = s_8(F) = 11221122$;

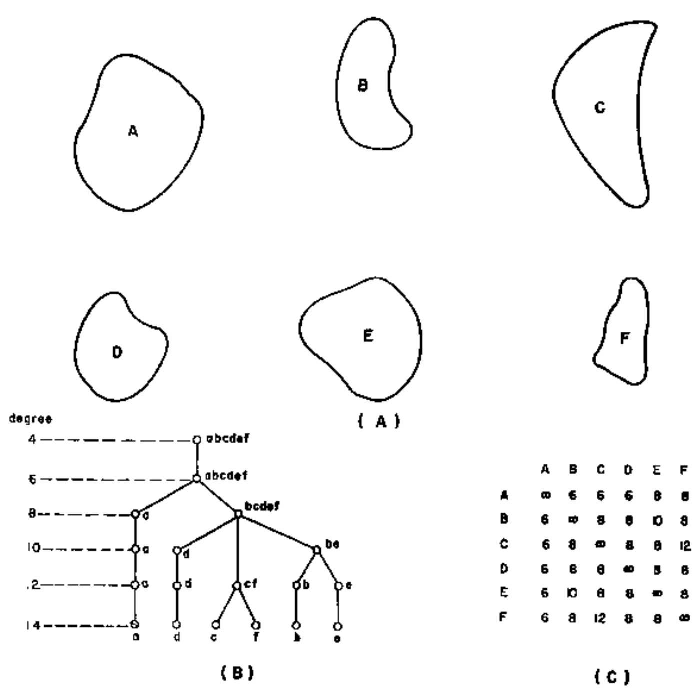
$s_{10}(A) = 1212212122$; $s_{10}(B) = 1121221123$;

$s_{10}(C) = 1122113113$;

$s_{10}(D) = s_{10}(E) = 1131212122$;

$s_{10}(F) = 1122121213$; $s_{12}(D) = 113113121213$;

$s_{12}(E) = 113121221213$.



	A	B	C	D	E	F
A	∞	6	6	6	8	8
B	6	∞	8	8	10	8
C	6	8	∞	8	8	12
D	6	8	8	∞	8	8
E	6	10	8	8	∞	8
F	6	8	12	8	8	∞

Fig. 9. Degree of similarity. (A) Regions to be analyzed. (B) Similarity tree for (A). (C) Similarity matrix for regions (A). The shapes form a hierarchy, a tree with root at degree = 4.

Therefore, A and B have a degree of similarity equal to 6: $A \approx_6 B$. Also, $A \approx_8 E$; $E \approx_6 B$; $D \approx_{10} E$; etc.

This is represented in the figure both as a similarity tree and as a similarity matrix.

The similarity matrix is symmetrical; in fact, it is easily proved that, for arbitrary regions A and B,

(1) (Thm.) The relation "A and B have degree k of similarity" (for a fixed k) is not an equivalence relation.

(2) (Thm.) The relation "A and B have degree of similarity of at least k" (for a fixed k) *is* an equivalence relation.

In fact, the equivalence classes of (2) for $k = 10$ are nine, since there are only nine discrete shapes of order 10.

Informally speaking, the size (power) of the magnifying lens that barely confuses two regions gives the degree of similarity between such regions.

The comparison procedure could be visualized as follows: A number (a shape number of high order) is associated to each one of two regions. If the numbers are equal, the regions have identical shape. If not, another pair of numbers (shape numbers of the next lower order) is deduced, and so on until we find that the two numbers coincide. The number of stages needed is an indication of the dissemblance of the two shapes.

3.2 *The distance between two shapes*

Distance. (definition) The distance between two shapes A and B is defined to be the inverse of their degree of similarity. $d(A,B) \triangleq 1/k$.

Then d is an ultradistance, obeying

$$d(A,A) = 0 \qquad (1)$$

$$d(A,B) \geq 0;$$

$$d(A,B) = 0 \text{ if and only if } A = B \qquad (2)$$

$$d(A,C) \leq \mathrm{Sup}[d(A,B),\ d(B,C)]. \qquad (3)$$

3.3 *Comments on this theory of shapes*

a. *No parsing is necessary.* To find the degree of similarity between A and B, their shape numbers are compared for *equality*. Two shape numbers of different order are incommensurable. Two shape numbers of the same order are either equal or different. If different, that is it. There is no need to compare "how close in shape they are". String matching[(2)] is not needed.

To find out the degree of similarity, a binary search is used. First see whether the shape numbers at order 8 are equal or not. Then compare the shape numbers at the highest required accuracy (say, 100). Then at the middle. Then at the middle of the remaining valid half. And so on.

b. *Intuitively satisfying.* Shape numbers are not invariant under reflexions (mirror images), skewing, or unequal expansion along the X and Y axes. These transformations alter what could be considered the intuitive shape of a figure. At the end of the paper a Theory "B" of shapes is presented, where the last transformation is allowed, i.e. a circle and an ellipse have the same Bshape number.

c. *Occasional loop in the similarity tree.* Due to noise or the 50 per cent requirement for quantization, and at low orders, a transitory divergence and then convergence in the shapes of two regions is sometimes observed, v.gr.,

$$s_8(A) = s_8(B)$$

$$s_{10}(A) \neq s_{10}(B)$$

$$s_{12}(A) = s_{12}(B)$$

$$s_{14}(A) \neq s_{14}(B)$$

$$s_{16}(A) \neq s_{16}(B)$$

$$\vdots$$

i.e. they were already different at order 10, but they are again equal at order 12 (however, only to separate soon forever). This still gives a unique number for the shape of a region, but makes the definition of degree of similarity less attractive, and the procedure to find it, unreliable. Only loops of size 2 (such as the example given) have been found, infrequently. These loops disappear in theory B.

d. *Non existent shape numbers.* Shape number of order n may occasionally not exist for a given figure, due for instance to symmetrical holes of type 1 in Fig. 8. This does not bother the similarity procedure, but it is a nuisance not to have that shape number.

e. *Quantization of the eccentricity.* The basic rectangles of order 12 have eccentricities equal to 1 (the square of 3 by 3), 2 (the rectangle of 4 by 2) and 5 (the rectangle of 5 by 1). For an object of eccentricity 1.6, one of these has to be used. An error is going to be committed in any case. There seems to be no way out of this.

A theory is now presented that has none of these problems.

3.4 *Theory "B" for Shape description*

To obtain this new theory, the current theory undergoes some changes:

1. Force the eccentricity of any region to be equal to one, by performing an unequal dilation of its axes. The only discrete Bshapes that now exist are those obtained from squares. All the rectangles have disappeared.
2. Do not go into depressions (1 in Fig. 8) with width smaller than the size of the side of the cell of the grid. This avoids degenerate shapes.

   That is, if a region is "scratched" by thin lines (thinner than the size of the grid) that belong to the background ignore them (act as if they were not there) or else, if they cannot be ignored, this theory "B" says that the size of the grid is inappropriate to describe such region, and that its Bshape does not exist at this order. Higher resolution is needed.
3. Let the depressions where the sticks do go in (because they are wider than Part 1 of Fig. 8)

generate *B*shape numbers having a number of (ternary) digits larger than the expected order. That is, do not correct the anomaly that these depressions cause. The perimeter of the *B*shapes no longer tells its order.

4. Eliminate the orders that are not powers of two. The only valid orders for *B*shape numbers are 4, 8, 16, 32, ... These numbers still indicate the perimeter of the basic square of the region.

The procedure is the following:

How to find the B*shape number of order* n

1. Find the basic rectangle of the region and convert it to a square. Declare that the *B*shape number does not exist if the region has necks (isthums) or depressions (channels, fjords) narrower than $4/n$ or $_{2}2-n$.
2. Make a grid by dividing the side of the basic square into $n/4$ equal parts.
3. Mark with a 1 each cell of the grid of step 2 that is more than 50 per cent contained in the region. The collection of grid squares containing a 1 form a discrete *B*shape.
4. Find the shape number of the discrete *B*shape of step 3, and give that as answer (even if it has more than n ternary digits).

The order n *of a* B*shape number* is four times the number of parts into which the side of the basic square was divided. It is also the perimeter (measured by the number of sticks) of the basic square.

It is no longer the perimeter of the discrete *B*shape, nor the number of ternary digits of the *B*shape number.

Given a shape, it is easy to derive its *B*shape number. An example is given in Fig. 11.

The degree of similarity between the *B*shapes of two regions is obtained as before. No change in the definition.

Downwards constructability. Given the *B*shape number of order n of a region, the *B*shape number of order $n/2$ can be deduced from it, by regrouping appropriate sets of 4 neighboring cells into a cell for the lower order. Therefore, if two regions have the same *B*shape number of order n, they will continue to have equal *B*shape numbers of smaller order, until they cease to exist. This gets rid of problem 3.3c of the former theory.

Upwards existence. If the *B*shape number of order n of a region exists, the existence of numbers for higher order is guaranteed, since the channels or narrow parts that could not split the shape at order n, will also be unable to split it at higher orders. This defeats problem 3.3d of the former theory.

Quantization of the eccentricity. Finally, problem 3.3e of the former theory is not present in theory B because all eccentricities are now equal to 1.

Some example of similarity comparison using theory *B* are given in Fig. 10.

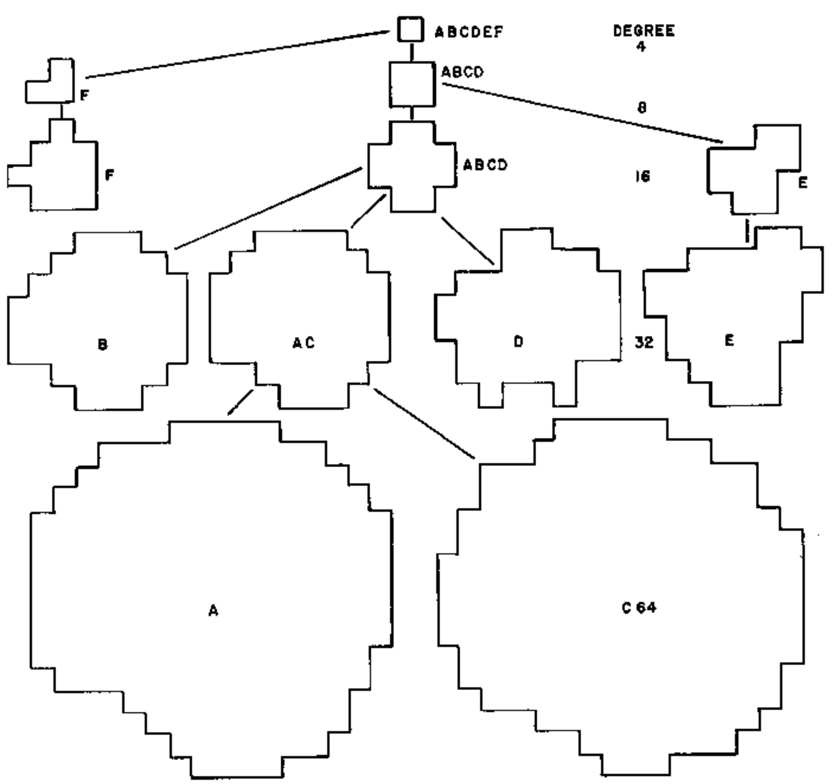
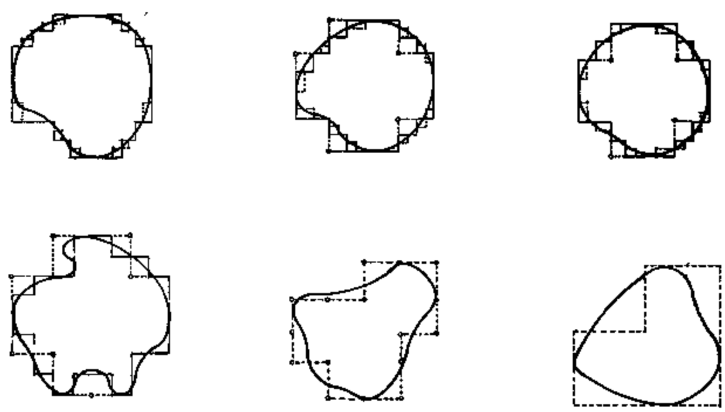


Fig. 10. Similarity tree for the *B*shapes of Regions *A* to *F*. The tree shows that the degree of similarity between *B* and *E* is 8, but between *B* and *C* is 16.



	A	B	C	D	E	F
A						
B	16					
C	32	16				
D	16	16	16			
E	8	8	8	8		
F	4	4	4	4	4	

TABLE "SIMILARITY MATRIX FOR THE BSHAPES OF REGIONS A–F"

Notice that $A \approx_{16} D$, $C \approx_{16} D$ but $E \approx_{32} C$

	A	B	C	D	E	F
A	0	1/16	1/32	1/16	1/8	1/4
B		0	1/16	1/16	1/8	1/4
C			0	1/16	1/8	1/4
D				0	1/8	1/4
E					0	1/4
F						0

TABLE "DISTANCE MATRIX FOR BSHAPES OF REGIONS A TO F"

A and C are very together (1/32) in Bshape

The region F is quite dissimilar (1/4) in shape to all others.

Fig. 11. Example of *B*shapes and their similarity. The arrows on the figures signal the beginning of the string of order 32 or 64.

Disadvantage of theory "B". Squeezing along one axis is now a valid (*B*shape preserving) transformation. Thus, either the application does not care about the eccentricity or aspect ratio, or this has to be carried as another parameter, in addition to the *B*shape number.

Also, more care needs to be exercised now when selecting the major and minor axis, to avoid noise perturbations. It may pay to use the rectangle suggested in.[3]

4. THE SHAPE NUMBERS OF SHAPES WITH HOLES

It is possible to assign shape numbers for regions with holes, and to use them for shape comparison and shape similarity measurement. The idea is to use the basic rectangle of the outer boundary for discretization of all the boundaries (both the outer and the inner boundaries). Using the shortest possible vertical or horizontal cuts, join the boundaries among them. Each cut reduces by one the number of boundaries. Finally, a single boundary is found. Then such a boundary can be described by an ordinary shape number. Such a shape number is then associated with the original region.

Notice that no other shape with holes could also be the owner of that shape number, since the new number has "touching edges" (those running along the cut). And since the set of cuts is unique (cf. discussion below), the resulting shape number is also unique. See Fig. 13.

The procedure is detailed now for *B*shapes. To find the *B*shape number of order n of a region with holes, proceed as follows:

1. Find the *B*shape number of order n of the outer boundary.
2. Using the grid defined in (1), find the discrete *B*shapes of the inner boundaries.
3. Let a "cut" be a sequence of purely vertical or purely horizontal segments of the grid. Find the minimum spanning tree of cuts that connects the boundaries (This tree can be found as follows: (a) find the two boundaries closest to each other; that is, the two boundaries with the shortest cut joining them. That cut belongs to the tree, and these two boundaries are now joined by such a cut. (b) Now, find the boundary closest to that new boundary. That defines another cut. This new cut belongs to the tree, and this new boundary is now joined (by such a cut) to the former collection of boundaries. (Now we have three boundaries joined by two cuts). (c) Keep iterating (b), each time adding a new boundary (the closest one) to the collection of boundaries, and its corresponding cut to the mini-

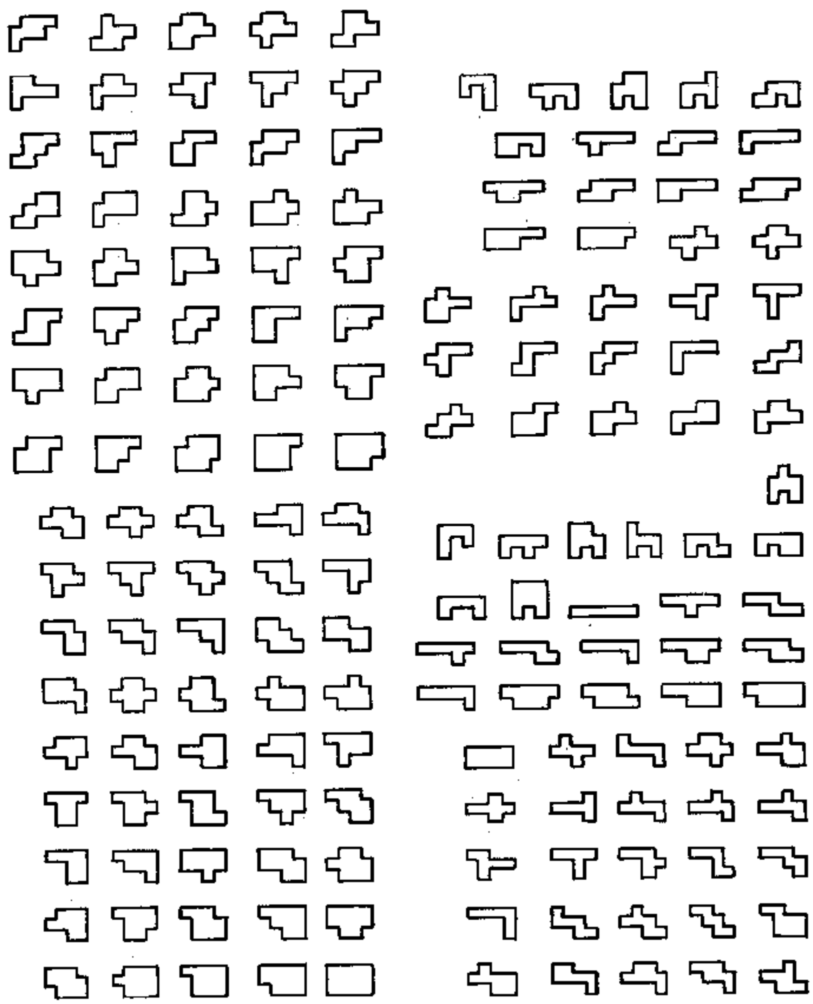


Fig. 12. All the shapes of order 14.

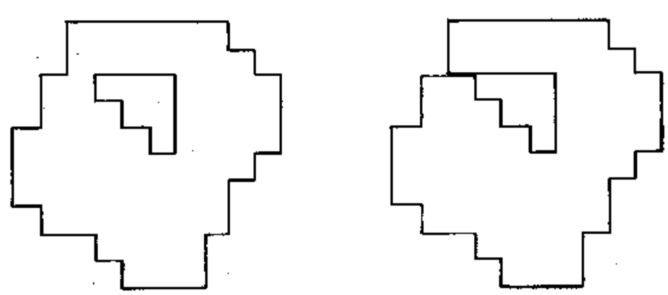


Fig. 13. Shapes with holes. To find the *B*shape number of a discrete shape (*A*) with a hole, cut a channe across the region, so as to have a simply connected region (*B*); then return the *B*shape of (*B*) as the answer The text provides more explanation.

mum spanning tree of cuts. When all boundaries are joined, stop.

The result is a simply connected boundary.

4. Find the *B*shape of this simply connected boundary, and give that as answer.

If there are two cuts of equal length, use the cut that minimizes the resulting *B*shape number. This favors cuts near the starting point of the *B*shape number.

With this tie-breaking rule, the *B*shape number is unique.

CONCLUSIONS

For each two-dimensional region, a shape number can be derived. This number depends exclusively on the form of the region.

These shape numbers can be found without table look-up or correlation or string matching.

The shape numbers can be of different *order*; high orders are more accurate for shape description. Informally, the number of ternary digits of a shape number will tell its order.

The *degree of similarity* between two regions, introduced in this paper, permits to find out how close in shape two regions are. Two regions with shapes that look alike will have a high degree of similarity.

Informally speaking, the degree of similarity between the shapes of two regions is the highest optical resolution (power of the magnifying lens) that still confuses them.

The *distance* between two shapes is defined and it is found to be an ultradistance or ultrametric.

The B*shape numbers* allow additional advantages and overcome some problems of the (ordinary) shape numbers.

The shape numbers of *figures with holes* are defined.

Suggestion for further work: apply the shape numbers to three-dimensional surfaces enclosing a volume.

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